$\{X_n\}$ MC, transition prob $\{P_{ij}\}$, with stat dist π,

$Y_n \triangleq X_{N-n}$ is MC with $\mathbb{P}(Y_{n+1}=j \mid Y_n=i) = \frac{\pi_j}{\pi_i} P_{ji}$.

$\{X_n\}$ reversible if $\{X\}$, $\{Y\}$ have same dynamics.

$$\Downarrow$$

Detailed Balance Condition

$$\pi_i P_{ij} = \pi_j P_{ji} \quad (\forall i, j)$$

Thm: μ satisfies DBC, then μ is stat meas.

Pf: $\sum_i \mu_i P_{ij} = \mu_j \sum_i P_{ji} = \mu_j$.

Only sufficient, not necessary.

DBC important for Markov Chain Monte Carlo (MCMC) methods, e.g., Metropolis-Hastings

e.g.: $P_{i,i+1} = 1 - \frac{i}{m}$, $P_{i,i-1} = \frac{i}{m}$ for $i \in \{0, 1, \dots, m\}$,

DBC: $\mu_i \cdot (1 - \frac{i}{m}) = \mu_{i+1} \frac{i+1}{m}$, $\frac{\mu_{i+1}}{\mu_i} = \frac{m-i}{i+1}$

$\frac{\mu_i}{\mu_0} = \frac{m!}{i!\,(m-i)!} = \binom{m}{i}$, set $\mu_0 = 1$, $\mu_i = \binom{m}{i}$

and $\sum_{i=1}^{m} \mu_i = 2^m$, so stat dist $\boxed{\pi_i = \frac{1}{2^m} \cdot \binom{m}{i}}$

e.g.: G finite connected graph, no loops and multiple edges, check DBC holds in equilibrium for RW on graph.

Pf: $\pi_i = \dfrac{d_i}{\sum_j d_j}$ and $P_{ij} = \dfrac{1}{d_i}$

so $\pi_i \cdot P_{ij} = \dfrac{1}{\sum_j d_j} = \pi_j \cdot P_{ji}$ ✓

Rem: along each undirected edge $(i) \text{---} (j)$ the symmetricity does not hold $\begin{cases} P_{ij} = \frac{1}{d_i} \\ P_{ji} = \frac{1}{d_j} \end{cases}$ when $d_i \neq d_j$.

If $P_{ij} = f(i,j)$, then $P_{ji} = f(j,i)$

If impose symmetricity, together with DBC for stat dist,

$$\begin{cases} d_i \cdot P_{ij} = d_j \cdot P_{ji} \\ P_{ij} = f(i,j), \ P_{ji} = f(j,i) \end{cases} \Rightarrow P_{ij} = \sqrt{\frac{d_j}{d_i}}$$

related to def of graph Laplacian!

For a review of cts-time Markov Chain, refer to the notes for 213A in fall 2023 on my website.

Poisson Process $\begin{cases} \text{memoryless arrival} \\ (\text{inter arrival times i.i.d. } \mathcal{E}(\lambda)) \\ \text{aggregation} \\ \text{thinning} \end{cases}$

☆ Key for Markov

pf see previous notes

discrete-state discrete-time MC $\{X_n\}$ $\Longrightarrow$ discrete-state cts-time MC $\{Y_t\}$

$(P_{ii} = 0 \quad \forall i)$

<u>Check</u>: $Y_t \triangleq X_{N_t}$ for Poisson process $\{N_t\}$ is a Markov Chain.

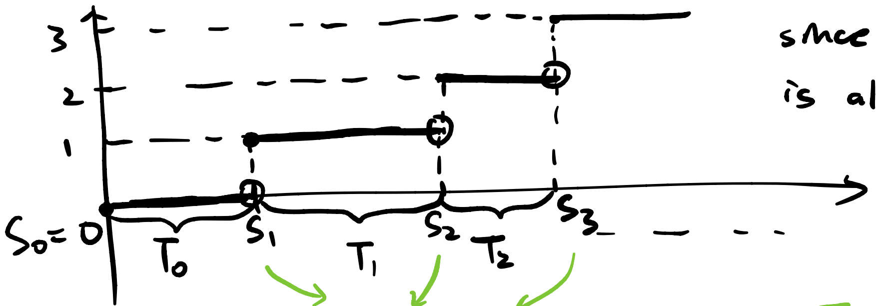
(intensity λ fixed)

$\{N\} \perp\!\!\!\perp \{X\}$

$$Y_{t_{n-1}} \longrightarrow Y_{t_n}$$

$\begin{cases} \text{transition time determined} \\ \text{state transition according to } \{X_n\} \end{cases}$

All interarrival times $T_0, T_1, \dots \overset{i.i.d.}{\sim} \mathcal{E}(\lambda)$.



since increment is always one.

arrival times, $S_k = T_0 + \dots + T_{k-1}$

$\{T_i\}$ also called "holding times" in CTMC

Then $Z_n \triangleq$ the value of X after n-th transition $= X_{S_n}$

so that $Y_t = \sum_{n=0}^{\infty} Z_n \cdot I_{[S_n, S_{n+1})}(t)$,

Lemma: (for Markov)

$$\mathbb{P}(Z_{n+1} = j, T_n > u \mid Z_0, \dots, Z_n, S_1, \dots, S_n)$$

$$= \mathbb{P}(Z_{n+1} = j, T_n > u \mid Z_0, \dots, Z_n, T_0, \dots, T_{n-1})$$

$$= \mathbb{P}\Big(Z_{n+1} = j, T_n > u \mid Z_n, T_0, \dots, T_{n-1} \;\Big|\; Z_0, \dots, Z_{n-1}\Big)$$

Markov $Z_{n+1}|_{Z_n} \perp\!\!\!\perp \{Z_0, \dots, Z_{n-1}\}$, $\{T_n\} \perp\!\!\!\perp \{Z_0, \dots, Z_{n-1}\}$

$$= \mathbb{P}(\underset{\uparrow \text{indep}}{Z_{n+1} = j}, \underset{\uparrow \text{indep}}{T_n > u} \mid Z_n, \underbrace{T_0, \dots, T_{n-1}}_{\uparrow \text{indep}})$$

$$= \mathbb{P}(Z_{n+1} = j, T_n > u \mid Z_n) = P_{Z_n, j} \cdot e^{-\lambda u}$$

a func only of Z_n

More generally, holding times only need to be independent, not i.i.d.

$$\Downarrow$$

$q(\cdot)$ as holding rate for each state

$$E_1, E_2, \dots \overset{i.i.d.}{\sim} \mathcal{E}(1), \quad T_i = \frac{E_i}{q(Z_i)}$$

The lemma still holds, proof modified to include E_i.

e.g.: $\{X_n\}$

$$\begin{tikzcd} i-1 & i \arrow[l, bend right, "P_{i,i-1}"'] \arrow[r, bend left, "P_{i,i+1}"] & i+1 \end{tikzcd}$$

given $Y_t = i$, B_i and D_i are time until next birth/death.

$$\underbrace{B_i \sim \mathcal{E}(\lambda_i), \quad D_i \sim \mathcal{E}(\mu_i)}_{\text{birth/death rates}}$$

holding rate $q_i = \lambda_i + \mu_i$, $\lambda_i = P_{i,i+1} q_i$, $\mu_i = P_{i,i-1} q_i$

since $T_i = \min\{B_i, D_i\} \sim \mathcal{E}(\lambda_i + \mu_i)$

e.g.: (cts-time branching with immigration)

Each particle indep. waits $\mathcal{E}(q)$ and either splits into 2 w.p. p or vanishes w.p. $1-p$. New particles immigrate into the system with Poisson arrival, intensity λ. $Y_t \triangleq$ # of particles in system at time t.

Pf:

Given $Y_t = i$, $L_1, \dots, L_i \overset{\text{iid}}{\sim} \mathcal{E}(q)$ are decision times and $I \sim \mathcal{E}(\lambda)$ is next arrival of immigrants.

Holding time $T_i = \min\{\underbrace{L_1, \dots, L_i}, I\} \sim \mathcal{E}(iq+\lambda)$

so $\underline{q_i = iq + \lambda}$

thinning w.p. p

Birth time $B_i = \min\{\qquad\qquad, I\} \sim \mathcal{E}(ipq+\lambda)$

so $\underline{\lambda_i = ipq + \lambda}$, $\underline{\mu_i = q_i - \lambda_i = i(1-p)q}$.

e.g.: (M/M/1 queue)

CTMC, BDC, $\begin{cases} \lambda_i = \lambda \\ \mu_i = \mu \\ q_i = \lambda + \mu \end{cases}$

e.g.: (M/M/∞ queue)

CTMC, BDC, $\begin{cases} \lambda_i = \lambda \\ \mu_i = i\mu \\ q_i = \lambda + i\mu \end{cases}$

Regularity of CTMC: $\forall i \in S,\ \mathbb{P}_i(S_\infty = \infty) = 1$

$$\Updownarrow$$

$$\forall i \in S,\ \mathbb{P}_i\left(\sum_n \frac{1}{q(X_n)} = \infty\right) = 1$$

e.g.: M/M/1 queue and Poisson process regular.